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Fractional Derivatives of Four Types of Fractional Trigonometric Functions

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***Abstract*: In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, the fractional differential problem of four types of fractional trigonometric functions is studied. We can obtain fractional derivative of any order of these four types of fractional trigonometric functions by using fractional geometric series. Moreover, our results are generalizations of traditional calculus results.**

***Keywords*: Jumarie type of R-L fractional derivative, new multiplication, fractional analytic functions, fractional trigonometric functions, fractional derivative, fractional geometric series.**

I. INTRODUCTION

Fractional calculus is an extension of ordinary calculus, which has a history of more than 300 years. Fractional calculus with any real or complex derivative and integral originated from Euler's work, even earlier than Leibniz's work. In recent years, the application of fractional calculus in many different fields such as physics, mechanics, mathematical economics, viscoelasticity, biology, control theory, and electrical engineering [1-13].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [14-18]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we mainly use fractional geometric series to find fractional derivative of any order of the following four types of fractional trigonometric functions:

$$[1 - r^{\alpha} cos_{\alpha}(\theta^{\alpha})] \otimes_{\alpha} [1 - 2r^{\alpha} cos_{\alpha}(\theta^{\alpha}) + r^{2\alpha}]^{\otimes_{\alpha}(-1)},$$

$$[r^{\alpha} sin_{\alpha}(\theta^{\alpha})] \otimes_{\alpha} [1 - 2r^{\alpha} cos_{\alpha}(\theta^{\alpha}) + r^{2\alpha}]^{\otimes_{\alpha}(-1)},$$

$$[1 + r^{\alpha} cos_{\alpha}(\theta^{\alpha})] \otimes_{\alpha} [1 + 2r^{\alpha} cos_{\alpha}(\theta^{\alpha}) + r^{2\alpha}]^{\otimes_{\alpha}(-1)},$$

$$[r^{\alpha} sin_{\alpha}(\theta^{\alpha})] \otimes_{\alpha} [1 + 2r^{\alpha} cos_{\alpha}(\theta^{\alpha}) + r^{2\alpha}]^{\otimes_{\alpha}(-1)}.$$

Where $0 < \alpha \leq 1$ and $|r^{\alpha}| < 1$. In fact, our results are generalizations of traditional calculus results.

II. PRELIMINARIES

Firstly, we introduce the fractional derivative used in this paper.

Definition 2.1 ([19])**:** Let $0 < \alpha \leq 1$, and θ_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{\theta_0}D_{\theta}^{\alpha}\right)[f(\theta)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\theta}\int_{\theta_0}^{\theta}\frac{f(t)-f(\theta_0)}{(\theta-t)^{\alpha}}dt\,, \tag{1}$$

where $\Gamma(\ )$ is the gamma function. On the other hand, for any positive integer p, we define $\left({}_{\theta_0}D_{\theta}^{\alpha}\right)^p[f(\theta)] = \left({}_{\theta_0}D_{\theta}^{\alpha}\right)\left({}_{\theta_0}D_{\theta}^{\alpha}\right)\cdots\left({}_{\theta_0}D_{\theta}^{\alpha}\right)[f(\theta)]$, the p-th order α-fractional derivative of $f(\theta)$.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([20])**:** *If* $\alpha, \beta, \theta_0, c$ *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{\theta_0}D_{\theta}^{\alpha}\right)\left[(\theta-\theta_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(\theta-\theta_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{\theta_0}D_{\theta}^{\alpha}\right)[c] = 0. \tag{3}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([21])**:** If θ, θ_0, and a_k are real numbers for all k, $\theta_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha}$ on some open interval containing θ_0, then we say that $f_\alpha(\theta^\alpha)$ is α-fractional analytic at θ_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([22])**:** Let $0 < \alpha \leq 1$, and θ_0 be a real number. If $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions defined on an interval containing θ_0,

$$f_\alpha(\theta^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(\theta-\theta_0)^{n\alpha}, \tag{4}$$

$$g_\alpha(\theta^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(\theta-\theta_0)^{n\alpha}\,. \tag{5}$$

Then we define

$$\begin{aligned} & f_\alpha(\theta^\alpha)\otimes_\alpha g_\alpha(\theta^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(\theta-\theta_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(\theta-\theta_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(\theta-\theta_0)^{n\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(\theta^\alpha)\otimes_\alpha g_\alpha(\theta^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{7}$$

Definition 2.5 ([23])**:** If $0 < \alpha \leq 1$, and θ is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(\theta^\alpha) = \sum_{n=0}^{\infty} \frac{\theta^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes_\alpha n}. \tag{8}$$

On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(\theta^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^k\theta^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes_\alpha 2n}, \tag{9}$$

and

$$sin_\alpha(\theta^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{10}$$

Definition 2.6 ([24])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ be two α-fractional analytic functions. Then $\left(f_\alpha(\theta^\alpha)\right)^{\otimes_\alpha n} = f_\alpha(\theta^\alpha)\otimes_\alpha \cdots \otimes_\alpha f_\alpha(\theta^\alpha)$ is called the nth power of $f_\alpha(\theta^\alpha)$. On the other hand, if $f_\alpha(\theta^\alpha)\otimes_\alpha g_\alpha(\theta^\alpha) = 1$, then $g_\alpha(\theta^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(\theta^\alpha)$, and is denoted by $\left(f_\alpha(\theta^\alpha)\right)^{\otimes_\alpha (-1)}$.

Definition 2.7 ([25])**:** Let $0 < \alpha \leq 1$, $i = \sqrt{-1}$, and $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$, $p_\alpha(\theta^\alpha)$, $q_\alpha(\theta^\alpha)$ be α-fractional real analytic at $\theta = \theta_0$. Let $z_\alpha(\theta^\alpha) = f_\alpha(\theta^\alpha) + ig_\alpha(\theta^\alpha)$ and $w_\alpha(\theta^\alpha) = p_\alpha(\theta^\alpha) + iq_\alpha(\theta^\alpha)$ be complex analytic at $\theta = \theta_0$. Define

$$\begin{aligned} & z_\alpha(\theta^\alpha)\otimes_\alpha w_\alpha(\theta^\alpha) \\ &= \left(f_\alpha(\theta^\alpha) + i\, g_\alpha(\theta^\alpha)\right)\otimes_\alpha \left(p_\alpha(\theta^\alpha) + i\, q_\alpha(\theta^\alpha)\right) \\ &= \left[f_\alpha(\theta^\alpha)\otimes_\alpha p_\alpha(\theta^\alpha) - g_\alpha(\theta^\alpha)\otimes_\alpha q_\alpha(\theta^\alpha)\right] + i\left[f_\alpha(\theta^\alpha)\otimes_\alpha q_\alpha(\theta^\alpha) + g_\alpha(\theta^\alpha)\otimes_\alpha p_\alpha(\theta^\alpha)\right]. \end{aligned} \tag{11}$$

Moreover, we define

$$|z_\alpha(\theta^\alpha)|_{\otimes_\alpha} = \left[z_\alpha(\theta^\alpha)\otimes_\alpha \overline{z_\alpha(\theta^\alpha)}\right]^{\otimes_\alpha\left(\frac{1}{2}\right)} = \left[[f_\alpha(\theta^\alpha)]^{\otimes_\alpha 2} + [g_\alpha(\theta^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha\left(\frac{1}{2}\right)}. \tag{12}$$

Definition 2.8 ([26])**:** The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(i\theta^\alpha)$.

III. MAIN RESULTS

In this section, we will find fractional derivative of any order of four types of fractional trigonometric functions. At first, three lemmas are needed.

Lemma 3.1 (fractional geometric series)**:** *If* $0 < \alpha \leq 1$ *and* $|z_\alpha(\theta^\alpha)|_{\otimes_\alpha} < 1$, *then*

$$[1 - z_\alpha(\theta^\alpha)]^{\otimes_\alpha (-1)} = \sum_{n=0}^{\infty}\left(z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha n}. \tag{13}$$

and

$$[1 + z_\alpha(\theta^\alpha)]^{\otimes_\alpha (-1)} = \sum_{n=0}^{\infty}(-1)^n\left(z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha n}. \tag{14}$$

Proof Since $|z_\alpha(\theta^\alpha)|_{\otimes_\alpha} < 1$, it follows that

$$\begin{aligned} & [1 - z_\alpha(\theta^\alpha)]\otimes_\alpha \sum_{n=0}^{\infty}\left(z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha n} \\ &= [1 - z_\alpha(\theta^\alpha)]\otimes_\alpha \lim_{m\to\infty} \sum_{n=0}^{m}\left(z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha n} \end{aligned}$$

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$$= \lim_{m\to\infty}\left[[1 - z_\alpha(\theta^\alpha)]\otimes_\alpha \sum_{n=0}^{m}\left(z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha n}\right]$$

$$= \lim_{m\to\infty}\left[1 - \left(z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha (m+1)}\right]$$

$$= 1\,.$$

Therefore,

$$[1 - z_\alpha(\theta^\alpha)]^{\otimes_\alpha (-1)} = \sum_{n=0}^{\infty}\left(z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha n}.$$

Similarly,

$$[1 + z_\alpha(\theta^\alpha)]\otimes_\alpha \sum_{n=0}^{\infty}(-1)^n\left(z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha n}$$

$$= [1 + z_\alpha(\theta^\alpha)]\otimes_\alpha \lim_{m\to\infty}\sum_{n=0}^{m}(-1)^n\left(z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha n}$$

$$= \lim_{m\to\infty}\left[[1 + z_\alpha(\theta^\alpha)]\otimes_\alpha \sum_{n=0}^{m}(-1)^n\left(z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha n}\right]$$

$$= \lim_{m\to\infty}\left[[1 - \left(-z_\alpha(\theta^\alpha)\right)]\otimes_\alpha \sum_{n=0}^{m}\left(-z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha n}\right]$$

$$= \lim_{m\to\infty}\left[1 - \left(-z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha (m+1)}\right]$$

$$= 1\,.$$

It follows that

$$[1 + z_\alpha(\theta^\alpha)]^{\otimes_\alpha (-1)} = \sum_{n=0}^{\infty}(-1)^n\left(z_\alpha(\theta^\alpha)\right)^{\otimes_\alpha n}\,. \qquad \text{Q.e.d.}$$

Lemma 3.2: *Let* $0 < \alpha \le 1$ *and* $|r^\alpha| < 1$, *then*

$$[1 - r^\alpha cos_\alpha(\theta^\alpha)]\otimes_\alpha [1 - 2r^\alpha cos_\alpha(\theta^\alpha) + r^{2\alpha}]^{\otimes_\alpha (-1)} = \sum_{n=0}^{\infty} r^{n\alpha} cos_\alpha(n\theta^\alpha). \tag{15}$$

And

$$[r^\alpha sin_\alpha(\theta^\alpha)]\otimes_\alpha [1 - 2r^\alpha cos_\alpha(\theta^\alpha) + r^{2\alpha}]^{\otimes_\alpha (-1)} = \sum_{n=0}^{\infty} r^{n\alpha} sin_\alpha(n\theta^\alpha). \tag{16}$$

Proof Let $z_\alpha(\theta^\alpha) = r^\alpha E_\alpha(i\theta^\alpha)$, then

$$|z_\alpha(\theta^\alpha)|_{\otimes_\alpha} = |r^\alpha E_\alpha(i\theta^\alpha)|_{\otimes_\alpha} = |r^\alpha| < 1. \tag{17}$$

By Lemma 3.1,

$$[1 - r^\alpha E_\alpha(i\theta^\alpha)]^{\otimes_\alpha (-1)} = \sum_{n=0}^{\infty}\left(r^\alpha E_\alpha(i\theta^\alpha)\right)^{\otimes_\alpha n}\,. \tag{18}$$

Hence,

$$\left[[1 - r^\alpha cos_\alpha(\theta^\alpha)] - i r^\alpha sin_\alpha(\theta^\alpha)\right]^{\otimes_\alpha (-1)} = \sum_{n=0}^{\infty} r^{n\alpha} E_\alpha(in\theta^\alpha)\,. \tag{19}$$

It follows that

$$\left[[1 - r^\alpha cos_\alpha(\theta^\alpha)] + i r^\alpha sin_\alpha(\theta^\alpha)\right]\otimes_\alpha \left[\left[[1 - r^\alpha cos_\alpha(\theta^\alpha)]^{\otimes_\alpha 2} + [r^\alpha sin_\alpha(\theta^\alpha)]^{\otimes_\alpha 2}\right]\right]^{\otimes_\alpha (-1)}$$

$$= \sum_{n=0}^{\infty} r^{n\alpha} cos_\alpha(n\theta^\alpha) + i\sum_{n=0}^{\infty} r^{n\alpha} sin_\alpha(n\theta^\alpha)\,. \tag{20}$$

Therefore,

$$\left[[1-r^{\alpha}cos_{\alpha}(\theta^{\alpha})]+ir^{\alpha}sin_{\alpha}(\theta^{\alpha})\right]\otimes_{\alpha}[1-2r^{\alpha}cos_{\alpha}(\theta^{\alpha})+r^{2\alpha}]^{\otimes_{\alpha}(-1)}$$
$$=\sum_{n=0}^{\infty}r^{n\alpha}cos_{\alpha}(n\theta^{\alpha})+i\sum_{n=0}^{\infty}r^{n\alpha}sin_{\alpha}(n\theta^{\alpha})\,. \quad (21)$$

Thus,

$$[1-r^{\alpha}cos_{\alpha}(\theta^{\alpha})]\otimes_{\alpha}[1-2r^{\alpha}cos_{\alpha}(\theta^{\alpha})+r^{2\alpha}]^{\otimes_{\alpha}(-1)}=\sum_{n=0}^{\infty}r^{n\alpha}cos_{\alpha}(n\theta^{\alpha}).$$

And

$$[r^{\alpha}sin_{\alpha}(\theta^{\alpha})]\otimes_{\alpha}[1-2r^{\alpha}cos_{\alpha}(\theta^{\alpha})+r^{2\alpha}]^{\otimes_{\alpha}(-1)}=\sum_{n=0}^{\infty}r^{n\alpha}sin_{\alpha}(n\theta^{\alpha}).$$ Q.e.d.

Lemma 3.3: *If* $0<\alpha\leq 1$ *and* $|r^{\alpha}|<1$, *then*

$$[1+r^{\alpha}cos_{\alpha}(\theta^{\alpha})]\otimes_{\alpha}[1+2r^{\alpha}cos_{\alpha}(\theta^{\alpha})+r^{2\alpha}]^{\otimes_{\alpha}(-1)}=\sum_{n=0}^{\infty}(-1)^{n}r^{n\alpha}cos_{\alpha}(n\theta^{\alpha}). \quad (22)$$

And

$$[r^{\alpha}sin_{\alpha}(\theta^{\alpha})]\otimes_{\alpha}[1+2r^{\alpha}cos_{\alpha}(\theta^{\alpha})+r^{2\alpha}]^{\otimes_{\alpha}(-1)}=-\sum_{n=0}^{\infty}(-1)^{n}r^{n\alpha}sin_{\alpha}(n\theta^{\alpha}). \quad (23)$$

Proof Let $z_{\alpha}(\theta^{\alpha})=r^{\alpha}E_{\alpha}(i\theta^{\alpha})$, then

$$|z_{\alpha}(\theta^{\alpha})|_{\otimes_{\alpha}}=|r^{\alpha}E_{\alpha}(i\theta^{\alpha})|_{\otimes_{\alpha}}=|r^{\alpha}|<1. \quad (24)$$

Using Lemma 3.1 yields

$$[1+r^{\alpha}E_{\alpha}(i\theta^{\alpha})]^{\otimes_{\alpha}(-1)}=\sum_{n=0}^{\infty}(-1)^{n}\left(r^{\alpha}E_{\alpha}(i\theta^{\alpha})\right)^{\otimes_{\alpha}n}. \quad (25)$$

Then

$$\left[[1+r^{\alpha}cos_{\alpha}(\theta^{\alpha})]+ir^{\alpha}sin_{\alpha}(\theta^{\alpha})\right]^{\otimes_{\alpha}(-1)}=\sum_{n=0}^{\infty}(-1)^{n}r^{n\alpha}E_{\alpha}(in\theta^{\alpha})\,. \quad (26)$$

Thus,

$$\left[[1+r^{\alpha}cos_{\alpha}(\theta^{\alpha})]-ir^{\alpha}sin_{\alpha}(\theta^{\alpha})\right]\otimes_{\alpha}\left[\left[[1+r^{\alpha}cos_{\alpha}(\theta^{\alpha})]^{\otimes_{\alpha}2}+[r^{\alpha}sin_{\alpha}(\theta^{\alpha})]^{\otimes_{\alpha}2}\right]\right]^{\otimes_{\alpha}(-1)}$$
$$=\sum_{n=0}^{\infty}(-1)^{n}r^{n\alpha}cos_{\alpha}(n\theta^{\alpha})+i\sum_{n=0}^{\infty}(-1)^{n}r^{n\alpha}sin_{\alpha}(n\theta^{\alpha})\,. \quad (27)$$

It follows that

$$\left[[1+r^{\alpha}cos_{\alpha}(\theta^{\alpha})]-ir^{\alpha}sin_{\alpha}(\theta^{\alpha})\right]\otimes_{\alpha}[1+2r^{\alpha}cos_{\alpha}(\theta^{\alpha})+r^{2\alpha}]^{\otimes_{\alpha}(-1)}$$
$$=\sum_{n=0}^{\infty}(-1)^{n}r^{n\alpha}cos_{\alpha}(n\theta^{\alpha})+i\sum_{n=0}^{\infty}(-1)^{n}r^{n\alpha}sin_{\alpha}(n\theta^{\alpha})\,. \quad (28)$$

Hence,

$$[1+r^{\alpha}cos_{\alpha}(\theta^{\alpha})]\otimes_{\alpha}[1+2r^{\alpha}cos_{\alpha}(\theta^{\alpha})+r^{2\alpha}]^{\otimes_{\alpha}(-1)}=\sum_{n=0}^{\infty}(-1)^{n}r^{n\alpha}cos_{\alpha}(n\theta^{\alpha}).$$

And

$$[r^{\alpha}sin_{\alpha}(\theta^{\alpha})]\otimes_{\alpha}[1+2r^{\alpha}cos_{\alpha}(\theta^{\alpha})+r^{2\alpha}]^{\otimes_{\alpha}(-1)}=-\sum_{n=0}^{\infty}(-1)^{n}r^{n\alpha}sin_{\alpha}(n\theta^{\alpha}).$$ Q.e.d.

Theorem 3.4: *Suppose that* $0<\alpha\leq 1$, $|r^{\alpha}|<1$, *and* p *is any positive integer, then*

$$\left({}_{0}D_{\theta}^{\alpha}\right)^{p}\left[[1-r^{\alpha}cos_{\alpha}(\theta^{\alpha})]\otimes_{\alpha}[1-2r^{\alpha}cos_{\alpha}(\theta^{\alpha})+r^{2\alpha}]^{\otimes_{\alpha}(-1)}\right]=\sum_{n=0}^{\infty}n^{p}r^{n\alpha}cos_{\alpha}\left(n\theta^{\alpha}+p\cdot\frac{T_{\alpha}}{4}\right), \quad (29)$$

$$\left({}_{0}D_{\theta}^{\alpha}\right)^{p}\left[[r^{\alpha}sin_{\alpha}(\theta^{\alpha})]\otimes_{\alpha}[1-2r^{\alpha}cos_{\alpha}(\theta^{\alpha})+r^{2\alpha}]^{\otimes_{\alpha}(-1)}\right]=\sum_{n=0}^{\infty}n^{p}r^{n\alpha}sin_{\alpha}\left(n\theta^{\alpha}+p\cdot\frac{T_{\alpha}}{4}\right), \quad (30)$$

$$\left({}_0D_\theta^\alpha\right)^p\left[[1+r^\alpha cos_\alpha(\theta^\alpha)]\otimes_\alpha[1+2r^\alpha cos_\alpha(\theta^\alpha)+r^{2\alpha}]^{\otimes_\alpha(-1)}\right]=\sum_{n=0}^{\infty}(-1)^n n^p r^{n\alpha}cos_\alpha\left(n\theta^\alpha+p\cdot\frac{T_\alpha}{4}\right), \quad (31)$$

and

$$\left({}_0D_\theta^\alpha\right)^p\left[[r^\alpha sin_\alpha(\theta^\alpha)]\otimes_\alpha[1+2r^\alpha cos_\alpha(\theta^\alpha)+r^{2\alpha}]^{\otimes_\alpha(-1)}\right]=-\sum_{n=0}^{\infty}(-1)^n n^p r^{n\alpha}sin_\alpha\left(n\theta^\alpha+p\cdot\frac{T_\alpha}{4}\right). \quad (32)$$

Proof By Lemma 3.2,

$$\left({}_0D_\theta^\alpha\right)^p\left[[1-r^\alpha cos_\alpha(\theta^\alpha)]\otimes_\alpha[1-2r^\alpha cos_\alpha(\theta^\alpha)+r^{2\alpha}]^{\otimes_\alpha(-1)}\right]$$
$$=\left({}_0D_\theta^\alpha\right)^p\left[\sum_{n=0}^{\infty}r^{n\alpha}cos_\alpha(n\theta^\alpha)\right]$$
$$=\sum_{n=0}^{\infty}n^p r^{n\alpha}cos_\alpha\left(n\theta^\alpha+p\cdot\frac{T_\alpha}{4}\right).$$

Similarly,

$$\left({}_0D_\theta^\alpha\right)^p\left[[r^\alpha sin_\alpha(\theta^\alpha)]\otimes_\alpha[1-2r^\alpha cos_\alpha(\theta^\alpha)+r^{2\alpha}]^{\otimes_\alpha(-1)}\right]$$
$$=\left({}_0D_\theta^\alpha\right)^p\left[\sum_{n=0}^{\infty}r^{n\alpha}sin_\alpha(n\theta^\alpha)\right]$$
$$=\sum_{n=0}^{\infty}n^p r^{n\alpha}sin_\alpha\left(n\theta^\alpha+p\cdot\frac{T_\alpha}{4}\right).$$

By Lemma 3.3, we have

$$\left({}_0D_\theta^\alpha\right)^p\left[[1+r^\alpha cos_\alpha(\theta^\alpha)]\otimes_\alpha[1+2r^\alpha cos_\alpha(\theta^\alpha)+r^{2\alpha}]^{\otimes_\alpha(-1)}\right]$$
$$=\left({}_0D_\theta^\alpha\right)^p\left[\sum_{n=0}^{\infty}(-1)^n r^{n\alpha}cos_\alpha(n\theta^\alpha)\right]$$
$$=\sum_{n=0}^{\infty}(-1)^n n^p r^{n\alpha}cos_\alpha\left(n\theta^\alpha+p\cdot\frac{T_\alpha}{4}\right).$$

And

$$\left({}_0D_\theta^\alpha\right)^p\left[[r^\alpha sin_\alpha(\theta^\alpha)]\otimes_\alpha[1+2r^\alpha cos_\alpha(\theta^\alpha)+r^{2\alpha}]^{\otimes_\alpha(-1)}\right]$$
$$=\left({}_0D_\theta^\alpha\right)^p\left[-\sum_{n=0}^{\infty}(-1)^n r^{n\alpha}sin_\alpha(n\theta^\alpha)\right]$$
$$=-\sum_{n=0}^{\infty}(-1)^n n^p r^{n\alpha}sin_\alpha\left(n\theta^\alpha+p\cdot\frac{T_\alpha}{4}\right).$$

Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions, we use fractional geometric series to find fractional derivative of any order of four types of fractional trigonometric functions. In addition, our results are generalizations of classical calculus results. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in applied mathematics and fractional differential equations.

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